J Glob Optim (2011) 49:15–21
DOI 10.1007/s10898-009-9521-0

Best proximity points: global optimal approximate solutions

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Received: 19 September 2009 / Accepted: 21 December 2009 / Published online: 5 January 2010

Abstract Let A and B be non-empty subsets of a metric space. As a non-self mapping $T : A \longrightarrow B$ does not necessarily have a fixed point, it is of considerable interest to find an element x in A that is as close to Tx in B as possible. In other words, if the fixed point equation $Tx = x$ has no exact solution, then it is contemplated to find an approximate solution x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. Indeed, best proximity point theorems investigate the existence of such optimal approximate solutions, called best proximity points, to the fixed point equation $Tx = x$ when there is no exact solution. As the distance between any element x in A and its image Tx in B is at least the distance between the sets A and B, a best proximity pair theorem achieves global minimum of $d(x, Tx)$ by stipulating an approximate solution x of the fixed point equation $Tx = x$ to satisfy the condition that $d(x, Tx) = d(A, B)$. The purpose of this article is to establish best proximity point theorems for *contractive* non-self mappings, yielding global optimal approximate solutions of certain fixed point equations. Besides establishing the existence of best proximity points, iterative algorithms are also furnished to determine such optimal approximate solutions.

Keywords Global optimal approximate solution · Fixed point · Best proximity point · Contractive mapping

1 Introduction

Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed linear spaces. Because a non-self mapping $T : A \longrightarrow B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best approximation theorems and best proximity point theorems are relevant in this perspective. A classical best approximation theorem, due

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to Ky Fan [8], states that if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X and $T : A \longrightarrow X$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. There have been many subsequent extensions and variants of Fan's Theorem, including those by Reich [13], Sehgal and Singh [17,18] and Prolla [12]. Further, Vetrivel et al. [20] have furnished a unified approach to such results.

On the other hand, though best approximation theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well. Indeed, if there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \longrightarrow B$, then it is desirable to find an approximate solution x such that $d(x, Tx)$ is minimum. In view of the fact that $d(x, Tx) \geq d(A, B)$, an absolute optimal approximate solution is an element x for which $d(x, Tx)$ attains the least possible value $d(A, B)$. As a result, a best proximity pair theorem offers sufficient conditions for the existence of an optimal approximate solution x, called a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(A, B)$. Interestingly, best proximity theorems also serve as a natural generalization of fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping.

Analysis of several variants of contractions for the existence of a best proximity point can be found in [1,4,6,9]. Best proximity point theorems for set valued mappings have been obtained in [2,3,10,11,14–16,19,21]. Further, Anthony Eldred et al. [5] have established best proximity point theorems for relatively non-expansive mappings.

The purpose of this article is to establish a best proximity point theorem for contractive non-self mappings in the setting of metric spaces, thereby producing an optimal approximate solution to the fixed point equation $Tx = x$ where $T : A \longrightarrow B$ is a contractive mapping. Interestingly, this result also generalizes Edelstein's fixed point theorem which states that a contractive self-mapping from a compact metric space to itself has a unique fixed point [7].

2 Contractive mappings

A key result on the existence of a best proximity point for contractive non-self mappings is furnished as follows.

Theorem 2.1 *Let A and B be non-empty compact subsets of a metric space. Suppose that the non-self mappings $T : A \longrightarrow B$ and $S : B \longrightarrow A$ satisfy the following conditions.*

(a) *T and S are contractive.*
(b) *$d(Tx, Sy) < d(x, y)$ whenever $d(x, y) > d(A, B)$ for $x \in A$ and $y \in B$.*
Then, there exist $x \in A$ and $y \in B$ such that

$$d(x, Tx) = d(A, B)$$
$$d(y, Sy) = d(A, B)$$
$$d(x, y) = d(A, B)$$

Further, for a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then, the sequence $\{x_{2n}\}$ converges to a best proximity point of T and the sequence $\{x_{2n+1}\}$ converges to a best proximity point of S.

If T or S has more than one best proximity point, then $d(A, B) > 0$ and hence the sets A and B are necessarily disjoint.

Proof Let $f : A \times B \longrightarrow R$ be defined as

$$f(x, y) = d(Tx, y) + d(Sy, x)$$

Since f is a continuous function on a compact set, it attains minimum at some element, say (x^*, y^*), in $A \times B$.

Suppose that Tx^* and y^* and are distinct.
$d(STx^*, Sy^*) < d(Tx^*, y^*)$ because S is contractive.
$d(TSy^*, Tx^*) \leq d(Sy^*, x^*)$ because T is contractive.
So, it follows from the preceding inequalities that

$$\begin{aligned} f(Sy^*, Tx^*) &= d(TSy^*, Tx^*) + d(STx^*, Sy^*) \\ &< d(Tx^*, y^*) + d(Sy^*, x^*) \\ &= f(x^*, y^*) \end{aligned}$$

which is contrary to the fact that f attains minimum at (x^*, y^*).
Therefore, $Tx^* = y^*$.
A similar argument can be given to show that $Sy^* = x^*$.
If $d(x^*, y^*) > d(A, B)$, then it follows from the condition (b) of the hypothesis that

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Sy^*) \\ &< d(x^*, y^*) \end{aligned}$$

which is a contradiction.
Therefore, $d(x^*, y^*) = d(A, B)$.

Hence, $d(x^*, Tx^*) = d(x^*, y^*) = d(A, B)$
$d(y^*, Sy^*) = d(x^*, y^*) = d(A, B)$.

On the other hand, because T and S are contractive,

$$\begin{aligned} d(x_{2n}, x^*) &= d(Sx_{2n-1}, Sy^*) \\ &\leq d(x_{2n-1}, y^*) \\ &= d(Tx_{2n-2}, Tx^*) \\ &\leq d(x_{2n-2}, x^*). \end{aligned}$$

Therefore, $\{d(x_{2n}, x^*)\}$ is a decreasing sequence and hence converges.
Since A is compact, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ converging to some element z in A.
Therefore, $d(x^*, z) = \lim d(x^*, x_{2n})$.
If x^* and z are distinct,

$$\begin{aligned} d(x^*, z) &> d(Tx^*, Tz) \\ &= d(y^*, Tz) \\ &\geq d(Sy^*, STz) \\ &= d(x^*, STz) \\ &= \lim d(x^*, STx_{2n_k}) \\ &= \lim d(x^*, x_{2n_k+2}) \\ &= d(x^*, z) \end{aligned}$$

which is a contradiction.
Thus, the sequence $\{x_{2n}\}$ converges to x^* which is a best proximity point of T.
Similarly, it can be shown that the sequence $\{x_{2n+1}\}$ converges to y^* which is a best proximity point of S.

If T has two distinct best proximity points x_1 and x_2, then it follows that

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, Tx_1) + d(Tx_1, Tx_2) + d(x_2, Tx_2) \\ &< 2d(A, B) + d(x_1, x_2) \end{aligned}$$

which necessitates that $d(A, B) > 0$ and hence the sets A and B are necessarily disjoint. Similarly, the same conclusion can be reached if S has two distinct best proximity points. This completes the proof of the theorem.

The following example illustrates the preceding theorem. It also brings out that uniqueness of the best proximity point is not plausible.

Example 2.2 Consider the space $C[0, \ \pi]$ with supremum norm.

For each α in $[-\frac{1}{2}, \ \frac{1}{2}]$, let $f_\alpha : [0, \ \pi] \longrightarrow R$ be defined as

$$f_\alpha(x) = \alpha \sin x.$$

For each β in $[0, \ 1]$, let $g_\beta : [0, \ \pi] \longrightarrow R$ be defined as

$$g_\beta(x) = \beta \sin x - 2.$$

Let $A = \{f_\alpha : -\frac{1}{2} \leq \alpha \leq \frac{1}{2}\}$ and $B = \{g_\beta : 0 \leq \beta \leq 1\}$.
Then, $d(A, B) = 2$.
Let the non-self mappings $T : A \longrightarrow B$ and $S : B \longrightarrow A$ be defined as

$$\begin{aligned} T(f_\alpha) &= f_{\alpha^2} - 2 \\ S(g_\beta) &= -\frac{1}{2}[g_{\beta^2} + 2] \end{aligned}$$

Then, all the hypotheses of the aforesaid theorem are satisfied.
Moreover, $d(f_\alpha, \ Tf_\alpha) = d(g_\beta, \ Sg_\beta) = d(f_\alpha, \ g_\beta) = d(A, \ B)$ whenever $-\frac{1}{2} \leq \alpha \leq 0$ and $0 \leq \beta \leq 1$.

The following common fixed point theorem for non-commuting contractive mappings is subsumed in Theorem 2.1.

Corollary 2.3 *Let A be a non-empty and closed subset of a compact metric space. Suppose that the mappings $T : A \longrightarrow A$ and $S : A \longrightarrow A$ satisfy the following conditions.*

(a) *T and S are contractive.*
(b) *$d(Tx, Sy) < d(x, y)$ whenever x and y are distinct elements in A.*

Then, T and S have a unique common fixed point.

The preceding result yields the following fixed point theorem, due to Edelstein [7], for contractive mappings.

Corollary 2.4 *If X is a compact metric space and $T : X \longrightarrow X$ is contractive, then T has a unique fixed point.*

Remark 2.5 The following example demonstrates that condition (b) in Theorem 2.1 is indispensable.

Consider the compact subsets $A = [0, \frac{1}{2}]$ and $B = [2, 3]$ of the space R.
Let $T : A \longrightarrow B$ be defined as

$$Tx = \frac{5}{2} + x^2.$$

Let $S : B \longrightarrow A$ be defined as

$$Sx = \frac{(x-2)^2}{2}.$$

Then, T and S are contractive.
But, the condition (b) of Theorem 2.1 does not hold good.
Further, it can be seen that T has no best proximity point.

Remark 2.6 The following simple example reveals that condition (a) in Theorem 2.1 is obligatory.

Consider the space R^2 with the metric defined as

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

$$\text{Let } A := \{(0, 0), (0, 1)\}$$
$$B := \{(1, 1), (1, 0)\}$$

Let $T : A \longrightarrow B$ and $S : B \longrightarrow A$ be defined as follows.

$$T(0, 0) = (1, 1)$$
$$T(0, 1) = (1, 0)$$
$$S(1, 1) = (0, 1)$$
$$S(1, 0) = (0, 0)$$

It can be observed that neither T is contractive nor S is contractive.
But, the condition (b) of Theorem 2.1 does hold good.
Moreover, it can be noted that T has no best proximity point.

3 An application to analytic functions of a complex variable

Theorem 3.1 *Let A and B be non-empty compact and convex subsets of a domain D of the complex plane. Let $f(z)$ and $g(z)$ be analytic functions in D. Suppose that*

(a) $f(A) \subseteq B$ *and* $g(B) \subseteq A$.
(b) $|f'(z)| < 1$ *for all* $z \in A$.
(c) $|g'(z)| < 1$ *for all* $z \in B$.
(d) $|f(z_1) - g(z_2)| < |z_1 - z_2|$.

whenever $|z_1 - z_2| > d(A, B)$ *for* $z_1 \in A$ *and* $z_2 \in B$.

Then, there exist $z_1 \in A$ *and* $z_2 \in B$ *such that*

$$|z_1 - f(z_1)| = d(A, B)$$
$$|z_2 - g(z_2)| = d(A, B)$$
$$|z_1 - z_2| = d(A, B).$$

Proof Since $|f'(z)|$ is a continuous function on the compact set A, it attains maximum at some point, say z_0, in A.

Let $k = |f'(z_0)|$.

Then, $k < 1$ and $|f'(z)| \leq k$ for all $z \in A$.

For distinct z and z' in A,

$$\begin{aligned} |f(z) - f(z')| &= \left| \int_{z'}^{z} f'(\xi) d\xi \right| \\ &\leq k|z - z'| \\ &< |z - z'| \end{aligned}$$

So, $f(z)$ is contractive. Similarly, it can be shown that $g(z)$ is contractive. Eventually, the result follows by invoking Theorem 3.1.

The following result for analytic functions is a special of the preceding theorem.

Corollary 3.2 *Let A be a non-empty, compact and convex subset of a domain D of the complex plane. Let $f(z)$ be an analytic function in D. Suppose that*

(a) $f(A) \subseteq A$.
(b) $|f'(z)| < 1$ *for all* $z \in A$.

Then, the equation $f(z) = z$ has at least one solution in A.

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